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Corner edge cutting and Dixon $\mathcal{A}$-resultant quotients

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Abstract

The classical Dixon resultant formulation gives the exact $\mathcal{A}$-resultant for the bi-degree rectangular monomial support. But the formulation gives a multiple of the $\mathcal{A}$-resultant when the monomial support is a bi-degree rectangular support with some corner edges removed. Fortunately, here the extraneous factors can be easily identified a priori and the $\mathcal{A}$-resultant can be expressed explicitly as a quotient: the determinant of a Dixon matrix divided by a product of brackets. All proofs in the paper are constructive. One of the proofs is mechanically done by a Maple program.

Keywords: Corner edge cutting; Dixon resultants; $\mathcal{A}$-resultants; Mechanical theorem proving

1. Introduction

Resultants are classical algebraic tools for determining whether a system of n polynomials in $n-1$ variables has a common root without actually finding the roots. Resultants have become very useful in widely diverse fields such as robotics, kinematics, computer aided geometric design, computer graphics, control theory and many others where polynomial systems arise.

The study of resultants goes back to the classical work of Bézout, Sylvester, Cayley, Macaulay, and Dixon. The resultant in quotient form given by Macaulay (1902, 1921) for the total degree configuration is not very helpful in many situations. For example, the Macaulay quotient vanishes in the bi-degree configuration due to a priori known roots at infinity and thus gives no information on roots of interest. This difficulty is overcome by the classical Dixon resultant which is customized for the bi-degree configuration. Efforts to

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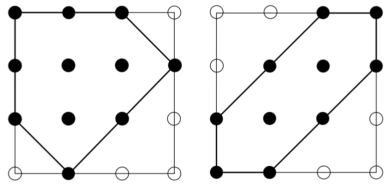

Fig. 1. A bi-cubic pentagonal monomial support and a bi-cubic hexagonal monomial support.

study the applicability of the Dixon formulation beyond the rectangular monomial support for the bi-degree configuration include Chionh (2001) and Chtcherba and Kapur (2002). This paper presents the discovery that when the monomial support of three bi-degree polynomials undergoes corner edge cutting the sparse resultant of the resulting unmixed monomial support $\mathcal{A}$ is the corresponding Dixon determinant divided by a product of at most four 3×3 determinants. While D'Andrea (2002) shows the construction of Macaulay-style sparse resultants, we give explicit Dixon-style sparse resultants for three bi-degree polynomials with an unmixed corner edge cut support. Our results differ from that of Chtcherba and Kapur (2002): they give algebraic conditions on the support of a generic unmixed bi-degree polynomial system for the Dixon determinant to be an exact sparse resultant and estimate the degree of the extraneous factor when the Dixon determinant is not exact, but we identify a class of supports called corner edge cutting supports and pinpoint the extraneous factors explicitly in the Dixon determinant for such supports. Other explicit sparse resultant expressions for three bi-degree polynomial equations are given by Zube (2000) and Zhang and Goldman (2000). For more discussion on classical resultants and $\mathcal{A}$-resultants, we refer the reader to Bikker and Uteshev (1999), Chionh and Goldman (1995), Cox et al. (1998), Emiris and Mourrain (1999), Gelfand et al. (1994), Kapranov et al. (1992), Kapur et al. (1994) and Saxena (1997).

An immediate application of the results is the implicitization (Chionh et al., 2000) of toric patches (Krasauskas, 2002). Toric patches whose supports are due to corner edge cutting arise easily, especially in the bi-cubic situation. For example, Fig. 1 shows a bi-cubic pentagonal monomial support and a hexagonal monomial support. Both supports are obtained from the bi-cubic rectangular support with corner cutting: the pentagonal support has rectangular cutting at the bottom-left, top-right corners and edge cutting at the bottom-right corner; the hexagonal support has edge cutting at the bottom-right and top-left corners.

The rest of the paper is organized into five sections. Section 2 provides the preliminaries. Section 3 states the main theorem of the paper and gives some illustrative examples. Section 4 recalls important results and developed intermediate results needed to prove the main theorem, which is then proved at the end of the section. Section 5 combines the results of rectangular corner cutting with corner edge cutting. We conclude in Section 6 with a summary and some open questions for further research.

2. Preliminaries

General terminology and notation and those pertinent to the Dixon formulation are defined in this section.

2.1. Indeterminates

We adopt the convention that $0^0 = 1$.

2.2. Sets

The set of integers is $\mathbf{Z}$. The cardinality of a set S is $\#S$. Set difference is written $A - B = \{x \in A \mid x \notin B\}$. If $B \subseteq A$, when we cut B from A we obtain $A - B$. We define $A - B_1 - \cdots - B_n = A - \cup_{i=1}^{n} B_i$. If $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ and $S \subseteq \mathbf{Z} \times \mathbf{Z}$ then $(a, b) + S = \{(a + x, b + y) \mid (x, y) \in S\}$.

For $a, b \in \mathbf{Z}$, the set of consecutive integers from a to b inclusively is denoted

$$a \mathinner{..} b = \{x \in \mathbf{Z} \mid a \le x \le b\}.$$

Note that $a \mathinner{..} b = \emptyset$ if $a > b$. If $a, b, c, d \in \mathbf{Z}$, the Cartesian product of two sets of consecutive integers is written

$$a \mathinner{..} b \times c \mathinner{..} d = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid a \le x \le b,\ c \le y \le d\}.$$

Note that $a \mathinner{..} b \times c \mathinner{..} d = \emptyset$ if $a > b$ or $c > d$.

We also abbreviate

$$a \mathinner{..} a \times c \mathinner{..} d = a \times c \mathinner{..} d,$$
$$a \mathinner{..} b \times c \mathinner{..} c = a \mathinner{..} b \times c.$$

2.3. Vectors, matrices, determinants, and brackets

The notation (i, j) denotes either an ordered pair or the row vector:

$$(i, j) = [a_{i,j}, b_{i,j}, c_{i,j}]. \tag{1}$$

This definition overloading causes no problems as the context will make the meaning clear. With "$\times$" denoting the vector cross product, we have

$$(i, j) \times (k, l) = \left[\begin{vmatrix} b_{i,j} & c_{i,j} \\ b_{k,l} & c_{k,l} \end{vmatrix}, \quad \begin{vmatrix} c_{i,j} & a_{i,j} \\ c_{k,l} & a_{k,l} \end{vmatrix}, \quad \begin{vmatrix} a_{i,j} & b_{i,j} \\ a_{k,l} & b_{k,l} \end{vmatrix} \right]. \tag{2}$$

The 3×3 determinant obtained by the vector triple product

$$(i, j) \cdot (k, l) \times (p, q) = \begin{vmatrix} a_{i,j} & b_{i,j} & c_{i,j} \\ a_{k,l} & b_{k,l} & c_{k,l} \\ a_{p,q} & b_{p,q} & c_{p,q} \end{vmatrix} \tag{3}$$

is called a bracket. Note that a bracket can also be obtained by the matrix multiplication of a 1×3 vector cross product and a 1×3 row vector:

$$(i, j) \cdot (k, l) \times (p, q) = ((k, l) \times (p, q))(i, j)^{\mathrm{T}} = (i, j)((k, l) \times (p, q))^{\mathrm{T}}.$$

For economy of space we abbreviate

$$(i, j) \cdot (k, l) \times (p, q) = ijklpq$$

in the examples.

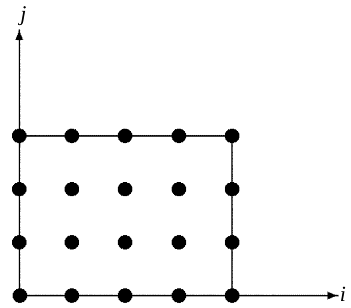

Fig. 2. The monomial support and the Newton polygon for a general bi-degree $(4, 3)$ polynomial.

2.4. Bi-degree polynomials, monomial supports

A polynomial $f(s, t)$ is bi-degree (m, n) in the variables (s, t) if its degrees in s and t are m and n respectively. That is, $f(s, t) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} s^i t^j$.

The monomial support of a polynomial $f(s, t)$ is the set of exponents (i, j) where the coefficient of the monomial $s^i t^j$ in f is non-zero. The monomial support of a general bi-degree (m, n) polynomial in (s, t) is thus

$$\mathcal{A}_{m,n} = 0 \mathinner{..} m \times 0 \mathinner{..} n. \tag{4}$$

Clearly $\mathcal{A}_{m,n}$ is a rectangular array in the (i, j) plane. The convex hull of the monomial support of a polynomial $f(s, t)$ is called the Newton polygon of f. For example, the monomial support and the Newton polygon of

$$f(s, t) = \sum_{i=0}^{4} \sum_{j=0}^{3} a_{i,j} s^i t^j; \quad a_{i,j} \neq 0, 0 \leq i \leq 4, 0 \leq j \leq 3$$

are shown in Fig. 2.

2.5. The classical bi-degree Dixon resultant

This section describes the construction of the classical Dixon resultant for three polynomial equations in two variables (Dixon, 1908).

Consider the three polynomials

$$f(s, t) = \sum_{(i,j)\in\mathcal{A}} a_{i,j} s^i t^j, \quad g(s, t) = \sum_{(i,j)\in\mathcal{A}} b_{i,j} s^i t^j, \quad h(s, t) = \sum_{(i,j)\in\mathcal{A}} c_{i,j} s^i t^j. \tag{5}$$

Their unmixed monomial support is $\mathcal{A} = \{(i, j) \mid \text{ some of } a_{i,j}, b_{i,j}, c_{i,j} \neq 0\} \subseteq \mathcal{A}_{m,n}$. We define the Dixon polynomial of f, g, h to be

$$\Delta_{\mathcal{A}}(f(s, t), g(\alpha, t), h(\alpha, \beta)) = \frac{1}{(s - \alpha)(t - \beta)} \begin{vmatrix} f(s, t) & g(s, t) & h(s, t) \\ f(\alpha, t) & g(\alpha, t) & h(\alpha, t) \\ f(\alpha, \beta) & g(\alpha, \beta) & h(\alpha, \beta) \end{vmatrix}. \tag{6}$$

Since the numerator vanishes when $s = \alpha$ or $t = \beta$, it is divisible by the denominator and $\Delta_{\mathcal{A}}$ is actually a polynomial in s, t, α, β. Our aim is to investigate the matrix form

$$\Delta_{\mathcal{A}} = \begin{bmatrix} \vdots \\ s^{\sigma} t^{\tau} \\ \vdots \end{bmatrix}^{\mathrm{T}} D_{\mathcal{A}} \begin{bmatrix} \vdots \\ \alpha^{a} \beta^{b} \\ \vdots \end{bmatrix} \tag{7}$$

where the coefficient matrix $D_{\mathcal{A}}$ and its determinant $|D_{\mathcal{A}}|$ is called the Dixon matrix and the Dixon determinant of the polynomials f, g, h respectively.

The monomials $s^{\sigma} t^{\tau}$ (or $\alpha^{a} \beta^{b}$) are called the row (or column) indices of $D_{\mathcal{A}}$. The monomial support $\mathcal{R}_{\mathcal{A}}$ (or $\mathcal{C}_{\mathcal{A}}$) of $\Delta_{\mathcal{A}}$ considered as a polynomial in s, t (or α, β) is called the row (or column) support of $D_{\mathcal{A}}$. That is,

$$\mathcal{R}_{\mathcal{A}} = \{(\sigma, \tau) \mid c_{\sigma,\tau,a,b} s^{\sigma} t^{\tau} \alpha^{a} \beta^{b} \text{ is a term in } \Delta_{\mathcal{A}} \text{ for some } a, b \text{ with } c_{\sigma,\tau,a,b} \neq 0\},$$

and

$$\mathcal{C}_{\mathcal{A}} = \{(a, b) \mid c_{\sigma,\tau,a,b} s^{\sigma} t^{\tau} \alpha^{a} \beta^{b} \text{ is a term in } \Delta_{\mathcal{A}} \text{ for some } \sigma, \tau \text{ with } c_{\sigma,\tau,a,b} \neq 0\}.$$

Since the numerator

$$\begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(\alpha,t) & g(\alpha,t) & h(\alpha,t) \\ f(\alpha,\beta) & g(\alpha,\beta) & h(\alpha,\beta) \end{vmatrix} = \sum_{(i,j),(k,l),(p,q)\in\mathcal{A}} (i,j)\cdot(k,l)\times(p,q) s^{i} t^{j+l} \alpha^{k+p} \beta^{q}, \tag{8}$$

the entries of $D_{\mathcal{A}}$ are linear in the coefficients of each of f, g, h.

Clearly $\Delta_{\mathcal{A}_{m,n}}$ is of degree $m-1$ in s, $2n-1$ in t, $2m-1$ in α, and $n-1$ in β. Consequently,

$$\mathcal{R}_{\mathcal{A}_{m,n}} = 0 \mathrel{..} m-1 \times 0 \mathrel{..} 2n-1, \qquad \mathcal{C}_{\mathcal{A}_{m,n}} = 0 \mathrel{..} 2m-1 \times 0 \mathrel{..} n-1, \tag{9}$$

and $\#\mathcal{R}_{\mathcal{A}_{m,n}} = \#\mathcal{C}_{\mathcal{A}_{m,n}} = 2mn$. Thus $D_{\mathcal{A}_{m,n}}$ is a square matrix of order $2mn$. The determinant $|D_{\mathcal{A}_{m,n}}|$ is the classical Dixon resultant of f, g, h.

For example,

$$|D_{\mathcal{A}_{1,1}}| = \begin{vmatrix} 100100 & 101100 \\ 110100 & 110110 \end{vmatrix}.$$

3. The Dixon $\mathcal{A}$-resultant quotients

This section introduces more notations and describes the main result as a theorem. Some of the notations are illustrated in Fig. 3.

Theorem 1. *Let the monomial support of f, g, h be*

$$\mathcal{A} = \mathcal{A}_{m,n} - E_1 - E_2 - E_3 - E_4$$

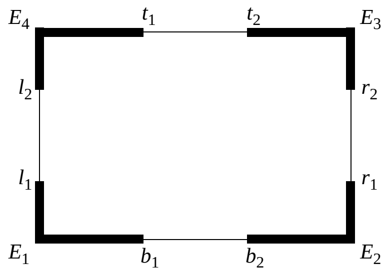

Fig. 3. Rectangular bi-degree (m, n) monomial support with position of corner edge points marked.

where E_1, E_2, E_3, E_4 each is a sequence of points on the edges of the bi-degree rectangular monomial support $\mathcal{A}_{m,n}$ such that

$$\begin{aligned} E_1 &= 0 \mathinner{..} b_1 \times 0 \cup 0 \times 0 \mathinner{..} l_1, \\ E_2 &= b_2 \mathinner{..} m \times 0 \cup m \times 0 \mathinner{..} r_1, \\ E_3 &= t_2 \mathinner{..} m \times n \cup m \times r_2 \mathinner{..} n, \\ E_4 &= 0 \mathinner{..} t_1 \times n \cup 0 \times l_2 \mathinner{..} n, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &-1 \le b_1 < b_1 + 1 \le b_2 - 1 < b_2 \le m + 1, \\ &-1 \le r_1 < r_1 + 1 \le r_2 - 1 < r_2 \le n + 1, \\ &-1 \le t_1 < t_1 + 1 \le t_2 - 1 < t_2 \le m + 1, \\ &-1 \le l_1 < l_1 + 1 \le l_2 - 1 < l_2 \le n + 1. \end{aligned} \tag{11}$$

The $\mathcal{A}$-resultant of f, g, h is then

$$\mathrm{Res}(\mathcal{A}) = \frac{|D_{\mathcal{A}}|}{B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}} \tag{12}$$

where $D_{\mathcal{A}}$ is the Dixon matrix for the monomial support $\mathcal{A}$; and

$$\begin{aligned} B_1 &= (0, l_1 + 1) \cdot (1, 1) \times (b_1 + 1, 0), \\ B_2 &= (b_2 - 1, 0) \cdot (m - 1, 1) \times (m, r_1 + 1), \\ B_3 &= (t_2 - 1, n) \cdot (m - 1, n - 1) \times (m, r_2 - 1), \\ B_4 &= (0, l_2 - 1) \cdot (1, n - 1) \times (t_1 + 1, n); \end{aligned} \tag{13}$$

$$\epsilon_1 = \begin{cases} 1 & \textit{if } b_1 \ge 1 \textit{ and } l_1 \ge 1, \\ 0 & \textit{otherwise}; \end{cases} \tag{14}$$

$$\epsilon_2 = \begin{cases} 1 & \textit{if } b_2 \le m - 1 \textit{ and } r_1 \ge 1, \\ 0 & \textit{otherwise}; \end{cases} \tag{15}$$

$$\epsilon_3 = \begin{cases} 1 & \textit{if } t_2 \le m - 1 \textit{ and } r_2 \le n - 1, \\ 0 & \textit{otherwise}; \end{cases} \tag{16}$$

$$\epsilon_4 = \begin{cases} 1 & \textit{if } t_1 \ge 1 \textit{ and } l_2 \le n - 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{17}$$

The convention $0^0 = 1$ and the exponents ϵ_i are needed to make $B_i^{\epsilon_i} = 1$ when the corner edge cutting E_i is degenerate. There are three types of corner edge cutting degeneracy: null (no cutting), the cutting is a corner point, the cutting is a vertical or a horizontal line.

The theorem statement may look complicated but its application is actually very straightforward. The following examples illustrate this ease of use with and without degeneracies. In the diagrams of the examples, elements of a monomial support are marked "•" and elements of edge E_i are marked i.

A detailed proof of the theorem is postponed to Section 4 because it is lengthy.

Example 1. Consider the monomial support $\mathcal{A}$

$$\begin{array}{c|ccccc}
t^4 & 4 & 4 & \bullet & 3 & 3 \\
t^3 & 4 & \bullet & \bullet & \bullet & 3 \\
t^2 & \bullet & \bullet & \bullet & \bullet & \bullet \\
t & 1 & \bullet & \bullet & \bullet & 2 \\
1 & 1 & 1 & \bullet & 2 & 2 \\
\hline
 & 1 & s & s^2 & s^3 & s^4
\end{array}$$

which is

$$\mathcal{A}_{4,4} - \{(0,0),(0,1),(1,0)\} - \{(3,0),(4,0),(4,1)\} \\ - \{(3,4),(4,3),(4,4)\} - \{(0,3),(0,4),(1,4)\}.$$

In this example the corner edges cut are non-degenerate. By Theorem 1, we have

$$\mathrm{Res}(\mathcal{A}) = \frac{|D_{\mathcal{A}}|}{B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}} = \frac{|D_{\mathcal{A}}|}{021120 \cdot 203142 \cdot 243342 \cdot 021324}.$$

Example 2. Consider the monomial support $\mathcal{A}$:

$$\begin{array}{c|cccc}
t^5 & \bullet & \bullet & 3 & 3 \\
t^4 & \bullet & \bullet & \bullet & 3 \\
t^3 & \bullet & \bullet & \bullet & 3 \\
t^2 & 1 & \bullet & \bullet & 3 \\
t & 1 & \bullet & \bullet & \bullet \\
1 & 1 & \bullet & \bullet & 2 \\
\hline
 & 1 & s & s^2 & s^3
\end{array}$$

which is

$$\mathcal{A}_{3,5} - \{(0,0),(0,1),(0,2)\} - \{(3,0)\} - \{(2,5),(3,2),(3,3),(3,4),(3,5)\} - \{\,\}.$$

Note that E_1 (a corner vertical line segment), E_2 (a corner point), and E_4 (null) are degenerate. By Theorem 1, we have $B_4 = 051405 = 0$ and

$$\mathrm{Res}(\mathcal{A}) = \frac{|D_{\mathcal{A}}|}{B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}} = \frac{|D_{\mathcal{A}}|}{031110^0 \cdot 202131^0 \cdot 152431^1 \cdot 0^0} = \frac{|D_{\mathcal{A}}|}{152431}.$$

4. A proof of the main theorem

This section consists of two parts. Sections 4.1–4.4 present important facts and develop intermediate results needed to prove the main theorem, which is proved in Sections 4.5 and 4.6.

4.1. A factorization theorem

The following theorem says that the Dixon matrix can be factored into a Sylvester matrix and a Bézout-like matrix.

Theorem 2. *The bi-degree* (m, n) *Dixon polynomial can be expressed as*

$$\Delta_{\mathcal{A}_{m,n}}(f(s,t), g(\alpha,t), h(\alpha,\beta)) = \begin{bmatrix} L^{\mathrm{T}} \\ \vdots \\ t^{n-1}L^{\mathrm{T}} \\ \vdots \\ s^{2m-1}L^{\mathrm{T}} \\ \vdots \\ s^{2m-1}t^{n-1}L^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} F \begin{bmatrix} 1 \\ \vdots \\ \beta^{n-1} \\ \vdots \\ \alpha^{2m-1} \\ \vdots \\ \alpha^{2m-1}\beta^{n-1} \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 1 \\ \vdots \\ t^{2n-1} \\ \vdots \\ s^{3m-1} \\ \vdots \\ s^{3m-1}t^{2n-1} \end{bmatrix}^{\mathrm{T}} SF \begin{bmatrix} 1 \\ \vdots \\ \beta^{n-1} \\ \vdots \\ \alpha^{2m-1} \\ \vdots \\ \alpha^{2m-1}\beta^{n-1} \end{bmatrix} \tag{19}$$

where $L = [f, g, h]$*; S is a* $6mn \times 6mn$ *Sylvester matrix, and F is a* $6mn \times 2mn$ *Bézout-like matrix.*

The entry of S indexed by $(s^\sigma t^\tau, s^a t^b)$*;* $0 \le \sigma \le 3m-1$, $0 \le \tau \le 2n-1$, $0 \le a \le 2m-1$, $0 \le b \le n-1$*; is*

$$S(s^\sigma t^\tau, s^a t^b) = \begin{cases} (\sigma - a, \tau - b) & \text{if } 0 \le \sigma - a \le m \text{ and } 0 \le \tau - b \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

The entry in F indexed by $(s^\sigma t^\tau, \alpha^a \beta^b)$*;* $0 \le \sigma \le 2m-1$, $0 \le \tau \le n-1$, $0 \le a \le 2m-1$, $0 \le b \le n-1$*; is*

$$F(s^\sigma t^\tau, \alpha^a \beta^b) = \sum_{k=\max(0,\sigma+1+a-m)}^{\min(m,\sigma+1+a)} \sum_{l=\max(0,\tau+1+b-n)}^{\min(\tau,b)} ((k,l) \times (\sigma+1+a-k, \tau+1+b-l))^{\mathrm{T}}. \tag{21}$$

The entry formula for the matrix F and a proof of the factorization are given in Chionh et al. (1999, 2002). The factorization is also proved in Chtcherba and Kapur (2000).

4.2. An entry formula for the Dixon matrix

Theorem 3. *The Dixon matrix entry indexed by* $(s^\sigma t^\tau, \alpha^a \beta^b)$ *is*

$$\begin{aligned} D(s^\sigma t^\tau, \alpha^a \beta^b) = & \sum_{u=0}^{\min(a,m-1-\sigma)} \sum_{v=0}^{\min(b,2n-1-\tau)} \sum_{k=\max(0,a-u-\sigma)}^{\min(m,a-u)} \sum_{l=\max(b+1,\tau+1+v-b)}^{\min(n,\tau+1+v)} B \\ & + \sum_{u=0}^{\min(a,m-1-\sigma)} \sum_{v=0}^{\min(b,2n-1-\tau)} \sum_{k=\max(0,a-u-m)}^{\min(\sigma,a-u)} \\ & \times \sum_{l=\max(b+1,\tau+1+v-n)}^{\min(n,\tau+v-b)} B \end{aligned} \tag{22}$$

where $B = (\sigma + 1 + u, \tau + 1 + v - l) \cdot (k, l) \times (a - u - k, b - v)$.

The entry formula is proved in Chionh (1997).

4.3. The row and column supports of $D_{\mathcal{A}}$, $\mathcal{A} = \mathcal{A}_{m,n} - E_1 - E_2 - E_3 - E_4$

Proposition 1. *Let* $\mathcal{A} = \mathcal{A}_{m,n} - E_1 - E_2 - E_3 - E_4$. *The row support of* $D_{\mathcal{A}}$ *is*

$$\mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}_{m,n}} - E_1 - [(-1, 0) + E_2] - [(-1, n - 1) + E_3] - [(0, n - 1) + E_4]$$

and the column support of $D_{\mathcal{A}}$ *is*

$$\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}_{m,n}} - E_1 - [(m - 1, 0) + E_2] - [(m - 1, -1) + E_3] - [(0, -1) + E_4].$$

The row and column supports theorem is proved in Chionh (2001). Briefly, the theorem says that if $\mathcal{A}$ is obtained by cutting corner edges E_1, E_2, E_3, E_4 from $\mathcal{A}_{m,n}$, then the row and column supports of $D_{\mathcal{A}}$ are obtained simply by cutting the corresponding corner edges from the row and column supports $\mathcal{R}_{\mathcal{A}_{m,n}}$ and $\mathcal{C}_{\mathcal{A}_{m,n}}$ (Fig. 4).

Note that to use the results in Chionh (2001) to describe the row and column supports of $D_{\mathcal{A}}$, the residual edges along the rectangular boundary after the cutting must be non-empty; this requirement is met by the conditions $b_1 + 1 \le b_2 - 1$, $r_1 + 1 \le r_2 - 1$, $t_1 + 1 \le t_2 - 1$, $l_1 + 1 \le l_2 - 1$ in Theorem 1.

4.4. Entries of the matrix F after top-right corner edge cutting

The following proposition shows that corner edge cutting simplifies the entries of the Bézout-like matrix F.

Proposition 2. *Let* $\mathcal{A} = \mathcal{A}_{m,n} - E_3$, $m \ge 1$, $n \ge 2$, $1 \le r_2 \le n - 1$, $1 \le t_2 \le m - 1$. *The entries in the three columns of* F *indexed by*

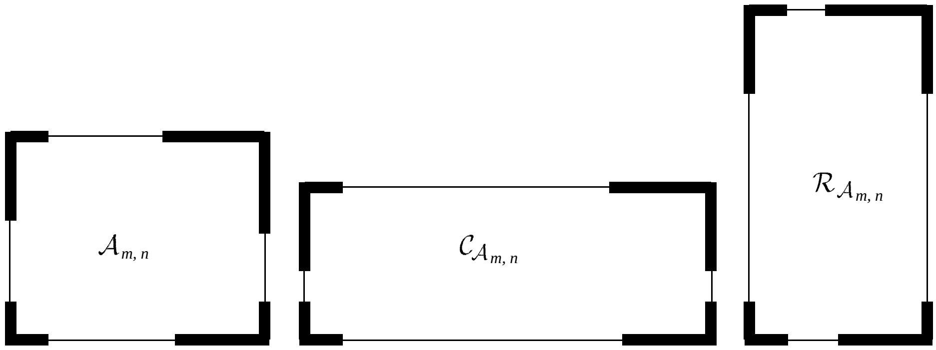

Fig. 4. The monomial support $\mathcal{A}$ (left), column support $\mathcal{C}_{\mathcal{A}}$ (center), and the row support $\mathcal{R}_{\mathcal{A}}$ (right) after corner edge cutting.

$$\alpha^{t_2+m-2}\beta^{n-1}, \alpha^{2m-2}\beta^{n-2}, \alpha^{2m-1}\beta^{r_2-2}$$

are as follows

$$F(s^\sigma t^\tau, \alpha^{t_2+m-2}\beta^{n-1}) = \begin{cases} ((m,\tau)\times(t_2-1,n))^{\mathrm{T}} & \sigma=0, 0\le\tau\le r_2-1, \\ 0 & 1\le\sigma\le 2m-1 \text{ or } \tau\ge r_2; \end{cases}$$

$$F(s^\sigma t^\tau, \alpha^{2m-2}\beta^{n-2}) = \begin{cases} ((m,\tau)\times(m-1,n-1))^{\mathrm{T}} & \sigma=0, 0\le\tau\le r_2-1, \\ 0 & 1\le\sigma\le 2m-1 \text{ or } \tau\ge r_2; \end{cases}$$

$$F(s^\sigma t^\tau, \alpha^{2m-1}\beta^{r_2-2}) = \begin{cases} ((m,\tau)\times(m,r_2-1))^{\mathrm{T}} & \sigma=0, 0\le\tau\le r_2-2, \\ 0 & 1\le\sigma\le 2m-1 \text{ or } \tau\ge r_2-1. \end{cases}$$

Note that column $\alpha^{2m-1}\beta^{r_2-2}$ exists provided $r_2\ge 2$.

Proof. Substituting $a=t_2+m-2$, $b=n-1$ into formula (21) and applying $t_2\ge 1$, the summation ranges become $k=\sigma+t_2-1\,..\,m$ and $l=\tau$. Thus,

$$F(s^\sigma t^\tau, \alpha^{t_2+m-2}\beta^{n-1}) = \sum_{k=\sigma+t_2-1}^{m} ((k,\tau)\times(\sigma+t_2+m-1-k,n))^{\mathrm{T}}.$$

To have $(\sigma+t_2+m-1-k,n)\notin E_3$, we need $\sigma+t_2+m-1-k\le t_2-1$. This forces $k=m$ and $\sigma=0$. To have $(k,\tau)=(m,\tau)\notin E_3$, we need $\tau\le r_2-1$. These constraints prove entry formula $F(s^\sigma t^\tau, \alpha^{t_2+m-2}\beta^{n-1})$.

Substituting $a=2m-2$, $b=n-2$ into formula (21) and applying $m\ge 1$, the summation ranges become $k=\sigma+m-1..m$ and $l=\max(0,\tau-1)..\min(\tau,n-2)$. When $\sigma\ge 2$ the summation range for k is empty, so

$$F(s^\sigma t^\tau, \alpha^{2m-2}\beta^{n-2}) = 0, \sigma\ge 2.$$

Observe that $\sigma=1$ forces $k=m$ and thus

$$F(st^\tau, \alpha^{2m-2}\beta^{n-2}) = \sum_{l=\max(0,\tau-1)}^{\min(\tau,n-2)} ((m,l)\times(m,\tau+n-1-l))^{\mathrm{T}}.$$

Since

$$\tau + n - 1 - l \geq \tau + n - 1 - \tau = n - 1 \geq r_2 \tag{23}$$

hence $(m, \tau + n - 1 - l) \in E_3$ and $F(st^\tau, \alpha^{2m-2}\beta^{n-2}) = 0$.

So we need only consider $\sigma = 0$ and the entries are

$$F(s^0t^\tau, \alpha^{2m-2}\beta^{n-2}) = \sum_{k=m-1}^{m} \sum_{l=\max(0,\tau-1)}^{\min(\tau,n-2)} ((k, l) \times (2m - 1 - k, \tau + n - 1 - l))^{\mathrm{T}}.$$

When $k = m - 1$, by inequality (23) we see that $(2m - 1 - k, \tau + n - 1 - l) \in E_3$ so the entry formula simplifies to

$$F(s^0t^\tau, \alpha^{2m-2}\beta^{n-2}) = \sum_{l=\max(0,\tau-1)}^{\min(\tau,n-2)} ((m, l) \times (m - 1, \tau + n - 1 - l))^{\mathrm{T}}.$$

By considering the three cases, $\tau = 0$, $1 \leq \tau \leq r_2 - 1$ and $\tau \geq r_2$, we prove the formula for $F(s^\sigma t^\tau, \alpha^{2m-2}\beta^{n-2})$.

The formula for $F(s^\sigma t^\tau, \alpha^{2m-1}\beta^{r_2-2})$ can be proved similarly. $\square$

4.5. *Divisibility by the brackets B_1, B_2, B_3, B_4*

First we show that

Proposition 3. *$|D_{\mathcal{A}_{m,n}-E_3}|$ is divisible by the bracket B_3.*

Proof. From the factorization theorem, we have

$$D(s^\sigma t^\tau, \alpha^a\beta^b) = \sum_{u=0}^{2m-1} \sum_{v=0}^{n-1} S(s^\sigma t^\tau, s^u t^v) F(s^u t^v, \alpha^a\beta^b)$$

where $0 \leq \sigma \leq m - 1$, $0 \leq \tau \leq 2n - 1$, $0 \leq a \leq 2m - 1$, and $0 \leq b \leq n - 1$. In Proposition 2, the formula involving F is non-zero only if $u = 0$ and $v \leq r_2 - 1$. When $r_2 \geq 2$, the entry in each of the three columns indexed by $\alpha^{t_2+m-2}\beta^{n-1}$, $\alpha^{2m-2}\beta^{n-2}$, $\alpha^{2m-1}\beta^{r_2-2}$ in the Dixon matrix becomes

$$\begin{aligned}
\sum_{v=0}^{r_2-1} S(s^\sigma t^\tau, s^0t^v)F(s^0t^v, \alpha^a\beta^b) &= \sum_{v=0}^{r_2-1} (\sigma, \tau - v) \cdot F(s^0t^v, \alpha^a\beta^b) \\
&= \sum_{v=0}^{r_2-1} (\sigma, \tau - v) \cdot ((m, v) \times W_{a,b}) \\
&= \sum_{v=0}^{r_2-1} (D_v(\sigma, \tau) \times P_v) \cdot W_{a,b} \qquad (24)
\end{aligned}$$

where $W_{a,b}$ is $X = (t_2 - 1, n)$, $Y = (m - 1, n - 1)$, or $Z = (m, r_2 - 1)$ respectively for the three columns $(a, b) = (t_2 + m - 2, n - 1)$, $(2m - 2, n - 2)$, $(2m - 1, r_2 - 2)$; $D_v(\sigma, \tau) = (\sigma, \tau - v)$; and $P_v = (m, v)$.

Thus we see that the brackets in any 3×3 submatrix of these three columns in the Dixon matrix are of this form:

$$\begin{vmatrix} (\sum_{i=0}^{r_2-1} A_i \times P_i) \cdot X & (\sum_{i=0}^{r_2-1} A_i \times P_i) \cdot Y & (\sum_{i=0}^{r_2-1} A_i \times P_i) \cdot Z \\ (\sum_{i=0}^{r_2-1} B_i \times P_i) \cdot X & (\sum_{i=0}^{r_2-1} B_i \times P_i) \cdot Y & (\sum_{i=0}^{r_2-1} B_i \times P_i) \cdot Z \\ (\sum_{i=0}^{r_2-1} C_i \times P_i) \cdot X & (\sum_{i=0}^{r_2-1} C_i \times P_i) \cdot Y & (\sum_{i=0}^{r_2-1} C_i \times P_i) \cdot Z \end{vmatrix}$$
$$= \begin{vmatrix} \sum_{i=0}^{r_2-1} A_i \times P_i \\ \sum_{i=0}^{r_2-1} B_i \times P_i \\ \sum_{i=0}^{r_2-1} C_i \times P_i \end{vmatrix} |X^T Y^T Z^T| \tag{25}$$

where the ordered pairs A_i, B_i, C_i are $D_i(\sigma, \tau)$ for the corresponding σ and τ. Clearly expression (25) is divisible by the bracket

$$|X^T Y^T Z^T| = B_3 = (t_2 - 1, n) \times (m - 1, n - 1) \cdot (m, r_2 - 1).$$

When $r_2 = 1$, there is no column indexed by $\alpha^{2m-1}\beta^{r_2-2}$. In this case we consider only the two columns indexed by $\alpha^{t_2+m-2}\beta^{n-1}$, $\alpha^{2m-2}\beta^{n-2}$. Here the Dixon entry formula simplifies to

$$\sum_{v=0}^{r_2-1} S(s^\sigma t^\tau, s^0 t^v) F(s^0 t^v, \alpha^a \beta^b) = ((\sigma, \tau) \times P_0) \cdot W_{a,b}.$$

Every 2×2 determinant of these two columns in the Dixon matrix is of the form

$$\begin{vmatrix} (A_i \times P_0) \cdot X & (A_i \times P_0) \cdot Y \\ (B_i \times P_0) \cdot X & (B_i \times P_0) \cdot Y \end{vmatrix} = |(A_i \times B_i) \cdot P_0| \, |X^T Y^T P_0^T|. \tag{26}$$

Since $P_0 = (m, 0)$, we have B_3 divides $|D_{\mathcal{A}_{m,n}-E_3}|$ in both cases. $\square$

The following propositions show that similar divisibility results hold at the bottom-left, bottom-right, and top-left corners respectively.

Proposition 4. *$|D_{\mathcal{A}_{m,n}-E_1}|$ is divisible by the bracket B_1.*

Proof. Let

$$[f', g', h'] = \sum_{i=0}^{m} \sum_{j=0}^{n} [a'_{i,j} s'^i t'^j, b'_{i,j} s'^i t'^j, c'_{i,j} s'^i t'^j]$$

where

$$a'_{i,j} = a_{m-i,n-j}, b'_{i,j} = b_{m-i,n-j} \text{ and } c'_{i,j} = c_{m-i,n-j}.$$

The monomial support of $[f', g', h']$ is easily seen to be $\mathcal{A}' = \mathcal{A}_{m,n} - E'_3$, where

$$E'_3 = m - b_1 .. m \times n \cup m \times n - l_1 .. n.$$

By Proposition 3, $(m-b_1-1, n)' \cdot (m-1, n-1)' \times (m, n-l_1-1)'$ divides $|D_{\mathcal{A}'}|$. However, $(m - b_1 - 1, n)' = (b_1 + 1, 0)$, $(m - 1, n - 1)' = (1, 1)$, $(m, n - l_1 - 1)' = (0, l_1 + 1)$.

So, $(b_1 + 1, 0) \cdot (1, 1) \times (0, l_1 + 1)$ divides $|D_{\mathcal{A}'}|$. That is B_1 divides $|D_{\mathcal{A}'}|$. But, $|D_{\mathcal{A}'}| = |D_{\mathcal{A}}|$ as shown below.

$$\Delta_{\mathcal{A}'}(f'(s',t'), g'(\alpha',t'), h'(\alpha',\beta')) = \begin{bmatrix} 1 \\ \vdots \\ t'^{2n-1} \\ \vdots \\ s'^{m-1} \\ \vdots \\ s'^{m-1}t'^{2n-1} \end{bmatrix}^{\mathrm{T}} D_{\mathcal{A}'} \begin{bmatrix} 1 \\ \vdots \\ \beta'^{n-1} \\ \vdots \\ \alpha'^{2m-1} \\ \vdots \\ \alpha'^{2m-1}\beta'^{n-1} \end{bmatrix}.$$

Substitute $s' = \frac{1}{s}, t' = \frac{1}{t}, \alpha' = \frac{1}{\alpha}, \beta' = \frac{1}{\beta}$, we have

$$\Delta_{\mathcal{A}}(f(s,t), g(\alpha,t), h(\alpha,\beta)) = \begin{bmatrix} s^{m-1}t^{2n-1} \\ \vdots \\ s^{m-1} \\ \vdots \\ t^{2n-1} \\ \vdots \\ 1 \end{bmatrix}^{\mathrm{T}} D_{\mathcal{A}'} \begin{bmatrix} \alpha^{2m-1}\beta^{n-1} \\ \vdots \\ \alpha^{2m-1} \\ \vdots \\ \beta^{n-1} \\ \vdots \\ 1 \end{bmatrix}.$$

Hence, $D_{\mathcal{A}'} = P D_{\mathcal{A}} P$ where $P = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \swarrow & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{bmatrix}$ and we conclude $|D_{\mathcal{A}'}| = |D_{\mathcal{A}}|$.

This proves that B_1 divides $|D_{\mathcal{A}}|$. □

Proposition 5. *$|D_{\mathcal{A}_{m,n}-E_2}|$ is divisible by the bracket B_2.*

Proof. The proof is similar to Proposition 4. The transformation used is

$$[f', g', h'] = \sum_{i=0}^{m}\sum_{j=0}^{n}[a'_{i,j}s^i t'^j, b'_{i,j}s^i t'^j, c'_{i,j}s^i t'^j]$$

where

$$a'_{i,j} = a_{i,n-j}, b'_{i,j} = b_{i,n-j} \text{ and } c'_{i,j} = c_{i,n-j}$$

and substitute $t' = \frac{1}{t}, \beta' = \frac{1}{\beta}$ in $\Delta_{\mathcal{A}'}(f'(s,t'), g'(\alpha,t'), h'(\alpha,\beta'))$. □

Proposition 6. *$|D_{\mathcal{A}_{m,n}-E_4}|$ is divisible by the bracket B_4.*

Proof. The proof is similar to Proposition 4. The transformation used is

$$[f', g', h'] = \sum_{i=0}^{m}\sum_{j=0}^{n}[a'_{i,j}s'^i t^j, b'_{i,j}s'^i t^j, c'_{i,j}s'^i t^j]$$

where

$$a'_{i,j} = a_{m-i,j}, b'_{i,j} = b_{m-i,j} \text{ and } c'_{i,j} = c_{m-i,j}$$

and substitute $s' = \frac{1}{s}, \alpha' = \frac{1}{\alpha}$ in $\Delta_{\mathcal{A}'}(f'(s',t), g'(\alpha',t), h'(\alpha',\beta))$. □

Theorem 4. *$|D_{\mathcal{A}_{m,n}-E_1-E_2-E_3-E_4}|$ is divisible by the product $B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}$.*

Proof. The result follows from the proofs above because the divisibility proof at each corner is independent of the situations at the other corners except when $l_2 - 1 = l_1 + 1$ or $r_2 - 1 = r_1 + 1$. When this happens, we note that only two columns are needed to produce the divisor B_i. To see this, we need only examine the top-right corner as the situation at the other corners can be reduced to that of the top-right corner.

When $r_2 - 1 = r_1 + 1$, (this includes the cases $r_1 = -1, r_2 = 1$ and $r_1 = n - 1$, $r_2 = n + 1$), Eq. (24) becomes

$$\sum_{v=r_1+1}^{r_2-1} (D_v(\sigma,\tau) \times P_v) \cdot W_{a,b} = (D_{r_2-1}(\sigma,\tau) \times (m, r_2 - 1)) \cdot W_{a,b}$$

because $P_v = (m, v) = 0$ unless $v = r_2 - 1$. But any 2×2 determinant of the form

$$\begin{vmatrix} (A_i \times P_v) \cdot X & (A_i \times P_v) \cdot Y \\ (B_i \times P_v) \cdot X & (B_i \times P_v) \cdot Y \end{vmatrix} = |(A_i \times B_i) \cdot P_v| \, |X^{\mathrm{T}} Y^{\mathrm{T}} P_v^{\mathrm{T}}|$$

is divisible by $|X^{\mathrm{T}} Y^{\mathrm{T}} P_v^{\mathrm{T}}| = B_3$ when $v = r_2 - 1$. □

Remarks. From the above proof and the fact that the entire left (or right) edge is removed from the column support when $l_2 - l_1 = 2$ (or $r_2 - r_1 = 2$), the number of columns responsible for the extraneous factors at a left corner can be expressed as follows:

$$\epsilon_i \min(l_2 - l_1, 3),$$

$i = 1, 4$. Furthermore, when $l_2 - l_1 = 2$ and $\epsilon_1 = \epsilon_4 = 1$, only five of the six columns are distinct because there is only one column index on the left edge of the column support. The expression and situation for the right corners are similar.

4.6. $\frac{|D_{\mathcal{A}}|}{B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}}$ is the $\mathcal{A}$-Resultant

Finally, we are ready to prove Theorem 1. By the theory of $\mathcal{A}$-resultants (Cox et al., 1998), we only have to show that $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4})$ has the correct degree in the polynomial coefficients and $|D_{\mathcal{A}}| \neq 0$ for general f, g, h. These are established by Theorem 5 and the propositions following it.

Theorem 5. *Consider n columns $C_1(x_1, \ldots, x_m), \ldots, C_n(x_1, \ldots, x_m)$, whose entries are polynomials in the variables $x_1, \ldots, x_m$. If after specializing the variables $x_1, \ldots, x_m$ to the values $\xi_1, \ldots, \xi_m$ respectively, the columns $C_1(\xi_1, \ldots, \xi_m), \ldots, C_n(\xi_1, \ldots, \xi_m)$ are linearly independent, then the given columns $C_1(x_1, \ldots, x_m), \ldots, C_n(x_1, \ldots, x_m)$ are linearly independent.*

Proof. We will prove the theorem by contradiction.

Assume there are polynomials $a_1(x_1, \ldots, x_m), \ldots, a_n(x_1, \ldots, x_m)$ not all zero such that

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_m) C_i(x_1, \ldots, x_m) = 0.$$

Replacing x_1 by ξ_1, we have

$$\sum_{i=1}^{n} a_i(\xi_1, x_2, \ldots, x_m) C_i(\xi_1, x_2, \ldots, x_m) = 0.$$

If not all $a_i(\xi_1, x_2, \ldots, x_m)$ are zero, the columns with $x_1 = \xi_1$ remain independent. Otherwise, $(x_1 - \xi_1)$ to some power divides $a_i(\xi_1, x_2, \ldots, x_m)$ for $i = 1, \cdots, n$. We divide $a_i(\xi_1, x_2, \ldots, x_m)$ by the highest possible power of $(x_1 - \xi_1)$, and obtain

$$\sum_{i=1}^{n} a_i^{(1)}(\xi_1, x_2, \ldots, x_m) C_i(\xi_1, x_2, \ldots, x_m) = 0$$

where not all the $a_i^{(1)}$ are zero.

This argument can be repeated and eventually we obtain

$$\sum_{i=1}^{n} a_i^{(m)}(\xi_1, \ldots, \xi_m) C_i(\xi_1, \ldots, \xi_m) = 0$$

where not all $a_1^{(m)}, \ldots, a_n^{(m)}$ are zero.

This contradicts the independence of the specialized columns. Thus the original columns must be independent. □

Proposition 7. *If $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4}) \neq 0$, it has the correct degree in the coefficients.*

Proof. Note that the entries of $D_{\mathcal{A}}$ and of the brackets B_i are linear in each of the coefficients of f, g, h. Thus we need only to show that the order of $D_{\mathcal{A}}$ minus $\sum_{i=1}^{4} \epsilon_i$ is equal to twice the area of the Newton polygon of $\mathcal{A}$.

By Proposition 1, the order of $D_{\mathcal{A}}$ is $2mn - \sum_{i=1}^{4} \#E_i$. By direct calculation, we see that when the corner edge E_i is cut, an area of size $(\#E_i + \epsilon_i)/2$ is chipped away from the rectangular monomial support $\mathcal{A}_{m,n}$. Since

$$\left(2mn - \sum_{i=1}^{4} \#E_i\right) - \sum_{i=1}^{4} \epsilon_i = 2\left(mn - \sum_{i=1}^{4} \frac{(\#E_i + \epsilon_i)}{2}\right),$$

we see that $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4})$ has the expected degree. □

Proposition 8. *The columns that produce the extraneous factors B_1, B_2, B_3, B_4 are linearly independent.*

Proof. Here we consider only the case when the extraneous factors are due to twelve columns, the cases when the extraneous factors are due to fewer columns can be treated similarly. By Theorem 5, we need only consider the support

$$\begin{aligned}\mathcal{A} = &\{(0, l_1+1), (1,1), (b_1+1, 0)\} \cup \{(b_2-1, 0), (m-1, 1), (m, r_1+1)\}\\ &\cup \{(m, r_2-1), (m-1, n-1), (t_2-1, n)\}\\ &\cup \{(t_1+1, n), (1, n-1), (0, l_2-1)\}\end{aligned}$$

of twelve monomials which are the vertices of the Newton polygon. This is because if these twelve columns for this special $\mathcal{A}$ are independent then they will be independent for a general $\mathcal{A}$.

Here the twelve columns producing the extraneous factors are ordered as

$$\begin{aligned}&\alpha^{b_1+1}, \alpha\beta, \beta^{l_1+1}, \alpha^{m+b_2-2}, \alpha^{2m-2}\beta, \alpha^{2m-1}\beta^{r_1+1},\\ &\alpha^{m+t_2-2}\beta^{n-1}, \alpha^{2m-2}\beta^{n-2}, \alpha^{2m-1}\beta^{r_2-2}, \alpha^{t_1+1}\beta^{n-1}, \alpha\beta^{n-2}, \beta^{l_2-2}.\end{aligned}$$

Consider the rows of $D_{\mathcal{A}}$ indexed and ordered as

$$\begin{aligned}&t^{l_1+1}, s^{b_1}t^{l_1}, s^{t_1}t^{n+l_2-2}, s^{m-1}t^{r_1+1}, s^{b_2-1}t^{r_1}, s^{t_2-1}t^{n+r_2-2},\\ &s^{m-1}t^{n+r_2-2}, s^{t_2-1}t^{n+r_2-1}, s^{b_2-1}t^{r_1+1}, t^{n+l_2-2}, s^{t_1}t^{n+l_2-1}, s^{b_1}t^{l_1+1}.\end{aligned}$$

This 12×12 submatrix will be shown to be a lower triangular matrix with non-zero diagonal entries:

$$\begin{aligned}&-B_1, B_1, -(0, l_1+1)\cdot(0, l_2-1)\times(t_1+1, n), -B_2, B_2, (t_2-1, n)\cdot(m, r_1+1)\\ &\quad\times(m, r_2-1), B_3, -B_3, (b_2-1, 0)\cdot(m, r_1+1)\times(m, r_2-1),\\ &\quad B_4, -B_4, -(0, l_1+1)\cdot(0, l_2-1)\times(b_1+1, 0).\end{aligned}$$

The fact that the 12×12 submatrix is lower triangular with the said diagonal entries can be verified mechanically by a Maple program with the assume facility for symbolic m and n. The program checks that the bracket $(a, b)\cdot(c, d)\times(e, f)$ is a possible entry if and only if $(a, b), (c, d), (e, f) \in \mathcal{A}$. The program then verifies that the bracket is indeed an entry if and only if u, v, k, l pertaining to a, b, c, d, e, f satisfy the Dixon entry formula summation bounds given in Theorem 3. □

Proposition 9. $|D_{\mathcal{A}}| \neq 0$.

Proof. It is known that any maximal minor of $D_{\mathcal{A}}$ is a multiple of the $\mathcal{A}$-resultant (Saxena, 1997; Emiris and Mourrain, 1999). Since the columns in Proposition 8 are independent, there is a maximal minor M containing these columns. By Proposition 8, the maximal minor M has the factors $B_1^{\epsilon_1}B_2^{\epsilon_2}B_3^{\epsilon_3}B_4^{\epsilon_4}$. So we can write $M = B_1^{\epsilon_1}B_2^{\epsilon_2}B_3^{\epsilon_3}B_4^{\epsilon_4}N$ for some polynomial N. It follows that N is a multiple of the $\mathcal{A}$-resultant. By Proposition 7, the degree of N in the coefficients of the polynomials f, g, h is at least

$$2mn - \sum_{i=1}^{4}\#E_i - \sum_{i=1}^{4}\epsilon_i.$$

That means the degree of M in the coefficients of each of the polynomials is at least

$$2mn - \sum_{i=1}^{4} \# E_i$$

which is the order of $D_{\mathcal{A}}$. This means M and $|D_{\mathcal{A}}|$ differ by a constant factor and thus $|D_{\mathcal{A}}|$ is non-zero. $\square$

5. Corner edge cutting and rectangular corner cutting

We observe that a degenerate edge can be taken to be a null rectangle, a point rectangle, or an edge rectangle. This suggests that we can combine the results from corner edge cutting and rectangular corner cutting. This observation is stated as a corollary which says that Theorem 1 still applies when either rectangular corner cutting or corner edge cutting occurs at any corner of the bi-degree rectangular monomial support. The proof of the corollary is similar to the proof of the main theorem and is thus omitted.

Corollary 1. *Let $\mathcal{A}_{m,n}$ be the rectangular bi-degree monomial support of the bi-degree (m, n) polynomials f, g, h. Let R_1, R_2, R_3 and R_4 be bottom-left, bottom-right, top-right, and top-left rectangular corners of $\mathcal{A}_{m,n}$ respectively. Let S_i be either R_i or E_i. Define γ_i to be the following*

$$\gamma_i = \begin{cases} 1 & \text{if } S_i = E_i, \\ 0 & \text{otherwise}. \end{cases} \tag{27}$$

Let $\mathcal{A} = \mathcal{A}_{m,n} - S_1 - S_2 - S_3 - S_4$. Then $\mathrm{Res}(\mathcal{A}) = \frac{|D_{\mathcal{A}}|}{B_1^{\gamma_1} B_2^{\gamma_2} B_3^{\gamma_3} B_4^{\gamma_4}}$.

Example 3. Consider the monomial support $\mathcal{A}$

$$\begin{array}{r|cccc} t^5 & \bullet & 3 & 3 & 3 \\ t^4 & \bullet & \bullet & \bullet & 3 \\ t^3 & \bullet & \bullet & \bullet & 3 \\ t^2 & 1 & 1 & \bullet & 3 \\ t & 1 & 1 & \bullet & 3 \\ 1 & 1 & 1 & \bullet & \bullet \\ \hline & 1 & s & s^2 & s^3 \end{array}$$

which is

$$\begin{aligned} &\mathcal{A}_{3,5} - \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\} - \{\,\} \\ &- \{(1,5), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5)\} - \{\,\}. \end{aligned}$$

By Corollary 1, we have $B_2 = 302130 = 0$ and $B_4 = 051405 = 0$.

$$\mathrm{Res}(\mathcal{A}) = \frac{|D_{\mathcal{A}}|}{B_1^{\gamma_1} B_2^{\gamma_2} B_3^{\gamma_3} B_4^{\gamma_4}} = \frac{|D_{\mathcal{A}}|}{031120^0 \cdot 0^0 \cdot 052430 \cdot 0^0} = \frac{|D_{\mathcal{A}}|}{052430}.$$

Note that $D_{\mathcal{A}_{m,n}}$ is a 30×30 matrix, and $D_{\mathcal{A}}$ is a 17×17 matrix.

6. Conclusion

For a bi-degree rectangular monomial support that has undergone corner edge cutting, the Dixon formulation for the $\mathcal{A}$-resultant has at most four extraneous factors. We have identified precisely these extraneous factors B_1, B_2, B_3, B_4, thereby determined exactly that the $\mathcal{A}$-resultant is $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4})$.

The proof consists of three components. Focusing on the top-right corner, we used the factorization theorem $D = S \cdot F$ to determine that three special columns of the Dixon matrix have a special property: any 3×3 submatrix in the identified columns has a factor B_3. Using the appropriate transformations, we proved the result for the other three corners. Following that, we checked that $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4})$ has the right degree if $|D_{\mathcal{A}}| \neq 0$ for general f, g, h. Finally, we confirmed that indeed $|D_{\mathcal{A}}| \neq 0$. Hence $|D_{\mathcal{A}}|/(B_1^{\epsilon_1} B_2^{\epsilon_2} B_3^{\epsilon_3} B_4^{\epsilon_4})$ is the $\mathcal{A}$-resultant.

We have noticed that large corner triangle cuttings of the monomial support result in $|D_{\mathcal{A}}| = 0$ leaving us clueless about the $\mathcal{A}$-resultant. We wonder if quadratic or higher degree extraneous factors can be found and the $\mathcal{A}$-resultant can be expressed in determinant quotient form.

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